# 3-D Transformations

3rd dimension added.
(Right Handed coordinate system)

① Translation in 3D is a simple extension from 2D

$$T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

if obj. is post multiplied

for pre multiplication of obj. matrix

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ dx & dy & dz & 1 \end{bmatrix}$$

② Scaling in 3D

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3x3 changes to 4x4)

③ 2-D rotation is just a 3D rotation abt. z axis.

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2-D rotation matrix

Rotation is always along origin i.e plane ⊥ to x & y i.e z axis

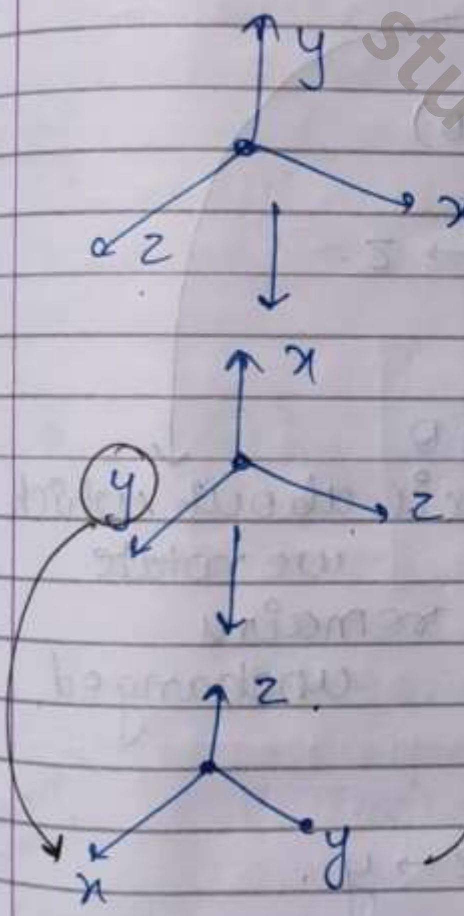
## 3D rotation
(rotation abt. x, y & z axis).

→ for rotation abt y axis. (rotation whole 3-D plane 2 times) $=$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↓

Put $y = z$
$x = z$
and $x = y$




Changing coordinates in anti-clockwise order.

→ Now rotation abt x axis.

⇓

$$\begin{bmatrix} \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Imp: Rotation about z

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

$z' = z$.

Rotation about y $R_y(\theta)$

$z \rightarrow y \rightarrow x \rightarrow z$

$z' = z\cos\theta - x\sin\theta$

$x' = z\sin\theta + x\cos\theta$

$y' = y$ } the axis about which we rotate remains unchanged.

Rotation about x axis

$y \rightarrow x \rightarrow z \rightarrow y$.

$x' = x$

$y' = y\cos\theta - z\sin\theta$

$z' = y\sin\theta + z\cos\theta$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Rotation } \underline{\text{abt. } x}$$

Theorem- In 3D, order of sequence of rotations matters

$$R_x \rightarrow R_y \neq R_y \rightarrow R_x$$

as matrix multiplication is non commutative.

Proof - $R_z(\alpha) R_y(\beta) = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} = [A]$

$R_y(\beta) R_z(\alpha) = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} = [B]$

We see that $[A] \neq [B]$.

Theorem:- $R_z(\alpha)$ and then $R_z(\beta)$

$$\Downarrow$$

(Proof can come) $R_z(\alpha + \beta)$

Reflection in 3-D

Reflection through x & y → Reflection through z axis.

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$\underbrace{x \text{ \& } y \text{ coordinate}}$

unchanged

$z$ coordinate → becomes -ve.

Reflection through xy plane = $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

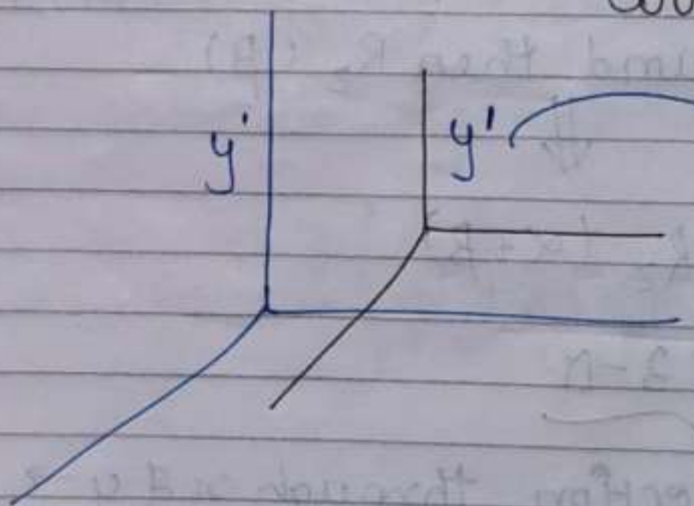
Reflection through xz plane. → y coordinate -ve
x & z unchanged.

Ref. through yz plane. → x coordinate -ve
y & z unchanged.

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



* Rotation abt an axis. parallel to coordinate axis.




① Translate y' so that it is coincides with y axis.

② Rotate

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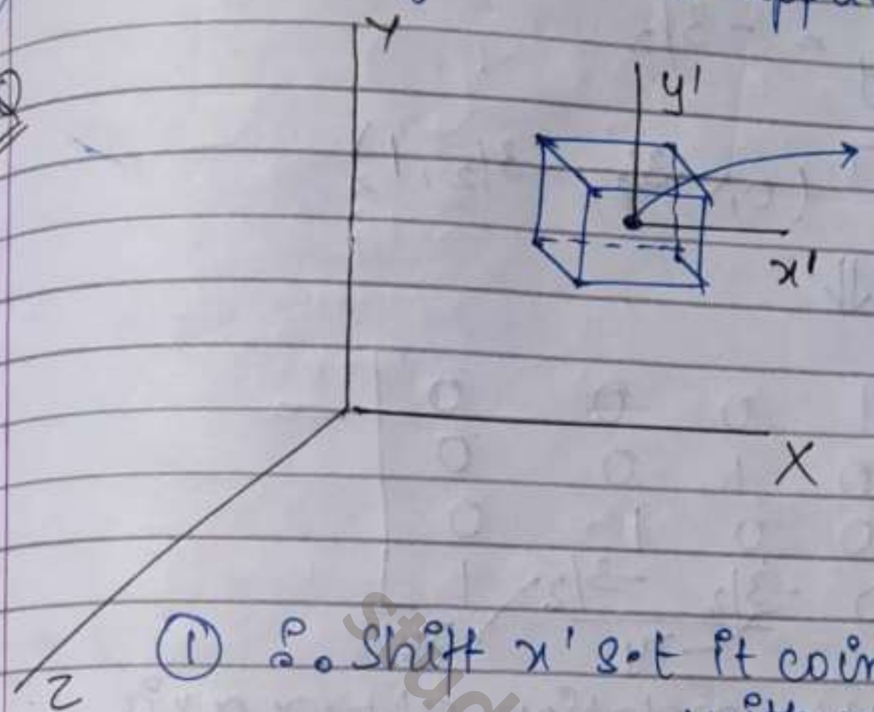
③ anti Transformations applied.




To find centroid of cube $\frac{x+y+z}{3}$

Rotate block by $\theta = 30^\circ$ abt. local $x'$ axis.

① i. Shift $x'$ s.t it coincides with $x$ axis.

②

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & 1 \end{array} \\ \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \end{array}$$

vertices of cube.

centroid of cube

$$\frac{12}{8}, \frac{12}{8}, \frac{12}{8}, \frac{8}{8}$$

$$= \left[\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1\right]$$

③ One method is to shift cube to origin

$$\left[-3/2, -3/2, -3/2, 1\right]$$

Or we can translate with y & z axis. s.t $x'$ coincides with $x$.

$\therefore T_x = 0 \qquad T_z = -3/2$

$T_y = -3/2$

$\therefore T_{(0,\ -3/2,\ -3/2,\ 1)}$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3/2 & -3/2 & 1 \end{bmatrix}$$

$T_x \qquad T_y \qquad T_z$

④ Now, apply rotation abt. x axis by angle of <u>30°</u>.

Rotation

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3/2 & -3/2 & & 1 \end{bmatrix} * \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Translation

in row vector format obj. is premultiplied.

$$* \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3/2 & 3/2 & 1 \end{bmatrix}}_{\text{Inverse Translation}}$$

$\therefore \underline{[O]\, T R T^{-1}}$

{ 2-D $\longrightarrow$ arbitrary axis
3-D $\longrightarrow$ arbitrary axis in space }

In 3D, first object must pass through origin. But that line is in air So, we first want that line to be in a plane $\therefore$ one additional step is rotation so that it comes in one of the planes.

additional step in 3-D

$\rightarrow$ If we coincide a line at x axis, we perform double rotation on axis other to coinciding axis. (i.e y & z)

$\rightarrow$ But how much rotation is needed?

* Rotation about an Arbitrary axis

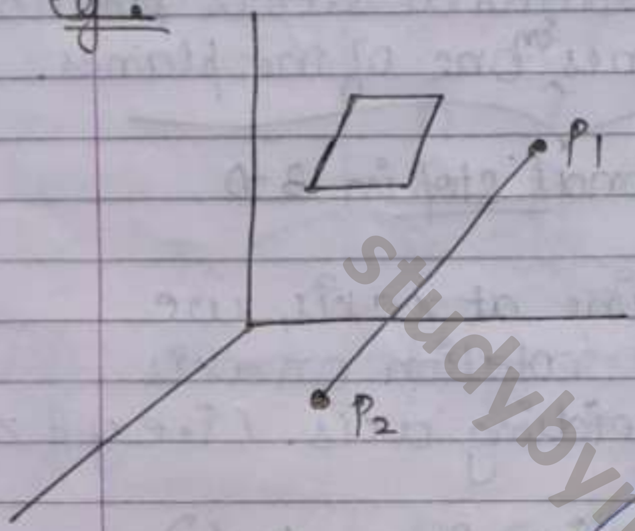
(1) Make axis P1P2 coincide with z axis
(2) Translation to move P1 to origin :-
T(-x1, -y1, -z1)
(3) Coincides one pt. of axis with origin.
(4) Rotation to coincide the shifted axis with z axis.
(5) R1 : rotation around X s.t axis lies on XZ plane.
(6) R2 : rotation around Y s.t axis coincides with z axis.
(7) R3 : rotate scene around z axis by an angle θ.
(8) Inverse transformations of R2, R1 & T1 to bring back axis to

original position.

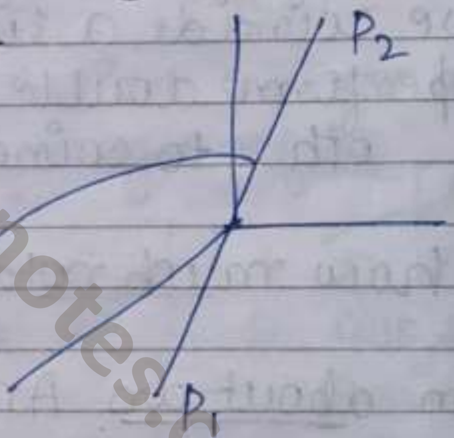
$$M = T^{-1} R_1^{-1} R_2^{-1} R_3 R_2 R_1 T.$$

actual rotation abt. z axis

eg:-




① Translate line




line vector coordinates

$P_2 - P_1$

$(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$

Now $\hat{u} = \dfrac{V}{|V|} = (a, b, c)$

unit vector along dir$^n$ of $P_1 P_2$.

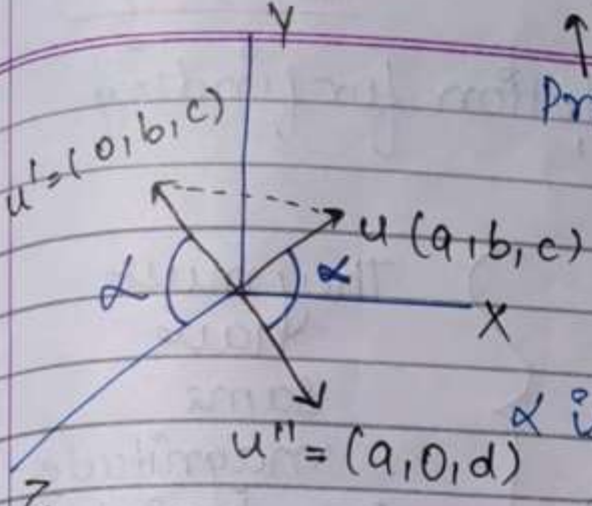
$$\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2}}$$

② Rotate u abt. X s.t it coincides with XZ plane.

coinciding end pts with plane

Project un on YZ plane $= u'(0,b,c)$

x coordinate becomes 0.




$\alpha$ is angle by $u'$ with z axis.

unit vector along $z \rightarrow (0,0,1)$
and $u' = (0,b,c)$

$$u' \cdot z = |u'|\,|z| \cos\theta.$$

$$c = \sqrt{b^2+c^2}\,(1)\cos\theta$$

$$\cos\theta = \cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

Also $u' \times z = |u'|\,|z| \sin\alpha$.

$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

NOTE:- We did projection of 'u' so as to calculate '$\alpha$' and we know that angle b/w $u'$ and z = angle b/w u & $u''$.

in rotation matrix, we have $\cos\theta$ & $\sin\theta$

rotation abt x axis

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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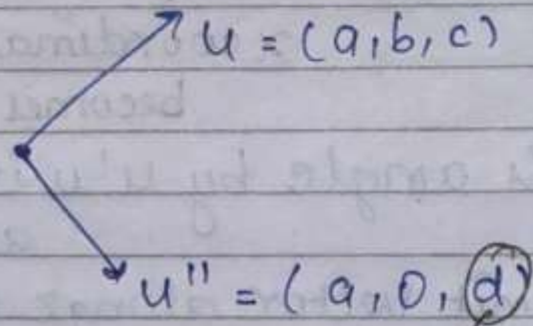
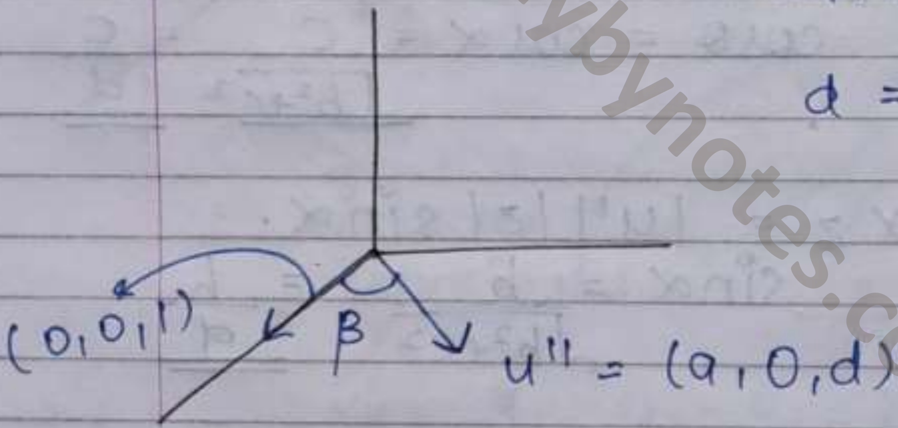
This was the 1st rotation for finding angle '$\alpha$'




They must have same magnitude i.e $\sqrt{a^2+b^2+c^2}$

$\therefore$ This becomes $\sqrt{b^2+c^2}$

- Rotate u'' abt y axis s.t it concides with z axis.




$d = (1)(\sqrt{a^2+d^2}) \cos\beta$

$\cos\beta = \dfrac{d}{\sqrt{a^2+d^2}}$

$= \dfrac{d}{\underbrace{\sqrt{a^2+b^2+c^2}}_{\text{mag. is 1}}} = d$

$\sin\beta = a$

$$\therefore R = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q Rotation matrices of rotation abt. x and y axis

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Rotate by θ abt z axis

$$M = T^{-1} R_1^{-1} R_2^{-1} R_3(\theta) R_2(\beta) R_1(\alpha) T$$

post multiplication of obj.

NOTE:- If obj. is pre multiplied, order will reverse.

(arbitrary)

- Coincide a plane with a planes given (XY, YZ, ZX)
- To do this, with a plane we have a normal ∴ we coincide normal with a ▇ ‖ axis. instead of coinciding plane.
- Coinciding normal with z axis coincides it with xy plane.

Reflection of plane through an arb. plane (neither XY, YZ or XZ)

* How to find coordinates of Normal ?

If in a plane we have 3 pts. coordinates of normal are cross product of adjacent edge vectors. Those normal planes can be used as coordinates of the axis.

$$[M] = [T][R_x][R_y]\underbrace{[R_f]}_{\text{reflection abt xy plane}}[R_y]^{-1}[R_x]^{-1}[T]^{-1}$$

# 3-D Shearing

**2D**

$$Sh_x = \begin{bmatrix} 1 & Sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + Sh_x \cdot y.$$

y coordinates change wrt. distance from $x$.

$$Sh_y = \begin{bmatrix} 1 & 0 & 0 \\ Sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$y' = y + Sh_y \cdot x.$$

In both dir$^n$ $\rightarrow$ both $x$ and $y$ coordinate change.

$$Sh_{xy} = \begin{bmatrix} 1 & Sh_x & 0 \\ Sh_y & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

**For 3D]**

$4 \times 4 \rightarrow$ Matrix $\rightarrow$ (Shearing of $x$ with be done wrt Dist. from $y$ and $z$).

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

# 3-D Shearing

**2D**

$$Shx = \begin{bmatrix} 1 & Shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = x + Shx \cdot y.$$

$$Shy = \begin{bmatrix} 1 & 0 & 0 \\ Shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

y coordinates change wrt. distance from x.

$$y' = y + Shy \cdot x.$$

In both dir$^n$ → both x and y coordinate change.

$$Shxy = \begin{bmatrix} 1 & Shx & 0 \\ Shy & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For 3D]

4x4 → matrix → (Shearing of x with be done wrt Dist. from y and z).

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & a & b & 0 \\ c & 1 & d & 0 \\ e & f & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$x' = x + ay + bz$ → shx wrt z.

$ay$ → shx wrt y

$y' = cx + y + dz$

$z' = ex + fy + z$

* **Translation** →
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ t_x & t_y & t_z & 1 \end{bmatrix}$$

* **Scaling** →
$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* **Rotation** →
$$\begin{bmatrix} \begin{bmatrix} \text{changes are} \end{bmatrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ \hline m & n & o & p \end{array}\right]$$

4x4 generalised matrix divided into 4 parts.

$[m \quad n \quad o]_{1\times 3}$ → responsible for Translation.

$$\left.\begin{array}{l}\text{scaling} \\ \text{Reflection} \\ \text{Rotation} \\ \text{Shearing.}\end{array}\right\} \quad \begin{bmatrix} a & b & c \\ e & f & g \\ i & j & k \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \rightarrow \text{scaling Transformation.}$$

$s$ → responsible for overall scaling.

$$\Updownarrow$$

$$\begin{bmatrix} 1/s & 0 & 0 & 0 \\ 0 & 1/s & 0 & 0 \\ 0 & 0 & 1/s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling factor → s.

If $s > 1$ → size reduces

If $s < 1$ → expanding size of element.

∴ [p] → responsible for overall scaling.

<u>Now</u>, last column.

$$\begin{bmatrix} d \\ h \\ l \\ p \end{bmatrix} \rightarrow \text{generally} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

1 → s for overall scaling

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All 3D Transformations are classified as Affine & Perspective Transformation.

(parallel lines remain parallel after transformation)

When last col values are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Whenever these values change acc. then known as perspective