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20 Feb, 19.

2-D Transformations

→ Slope of Transformed line. (Proof-2)

Parallel lines remain ∥ upon transformation i.e slope remains same.

(x_1, y_1) and (x_2, y_2)
↳ slope $= m = \frac{y_2 - y_1}{x_2 - x_1}$

And slope of transformed line is
↓

$$m^* = \frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{(bx_2 + dy_2) - (bx_1 + dy_1)}{(ax_2 + cy_2) - (ax_1 + cy_1)}$$

Transformed coordinates

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{b(x_2 - x_1) + d(y_2 - y_1)}{a(x_2 - x_1) + c(y_2 - y_1)} \quad \{\text{Rearranging}\}$$

$$= \frac{b + d\frac{(y_2 - y_1)}{(x_2 - x_1)}}{a + c\frac{(y_2 - y_1)}{(x_2 - x_1)}} \quad \{\text{divide by } (x_2 - x_1) \text{ throughout}\}$$

Transformed slope $= \boxed{\frac{b + dm}{a + cm}}$ —— (i)

m^* is thus independent of x_1, y_1, x_2, y_2 (Transformed coordinates)

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∴ A set of parallel lines will remain parallel after transformation.

eg:- $L1$ m_1 ⇓ L_2 m_2

For ∥ lines $m_1 = m_2$

$m_1^* = \frac{b + d m_1}{a + c m_1}$ $m_2^* = \frac{b + d m_2}{a + c m_2}$

since a, b, c, d are same & $m_1 = m_2$

∴ As long as previous slope & transformation matrix is same, transformed lines will have same slope.

If we transform all coordinates of a ∥gm, parallel lines will remain parallel but shape may change.

Transformation of Intersecting lines.

Do they intersect after Transformation.

Do they intersect at same/diff. angles

consider 2 intersecting lines

$y = m_1 x + b_1$

$y = m_2 x + b_2$

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Writing in matrix ↓.

$$y - m_1 x = b_1$$
$$y - m_2 x = b_2$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & -m_1 \\ 1 & -m_2 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix}$$

(col. vector form)

$$[x \;\; y] \begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix} = [b_1 \;\; b_2]$$

Obj. matrix — $[x \;\; y]$

Transformation matrix — $\begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix}$

gives row vector.

$$X m = B$$
$$X = B m^{-1}$$

$$m^{-1} = \frac{\text{Adj } m}{|m|}$$

$$|m| = -m_1 + m_2$$

$$\text{Adj } m = \begin{bmatrix} +1 & +m_2 \\ -1 & -m_1 \end{bmatrix}$$

Interchange main diagonal elements & reverse sign for other elements.

$$m^{-1} = \begin{bmatrix} \frac{1}{m_2 - m_1} & \frac{m_2}{m_2 - m_1} \\ \frac{-1}{m_2 - m_1} & \frac{-m_1}{m_2 - m_1} \end{bmatrix}$$

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$$X = [b_1 \quad b_2]_{(1\times2)} \begin{bmatrix} \frac{1}{m_2 - m_1} & \frac{m_2}{m_2 - m_1} \\ \frac{-1}{m_2 - m_1} & \frac{-m_1}{m_2 - m_1} \end{bmatrix}_{(2\times2)}$$

$$[x_i \quad y_i] = \underbrace{\begin{bmatrix} \frac{b_1 - b_2}{m_2 - m_1} & \frac{b_1 m_2 - b_2 m_1}{m_2 - m_1} \end{bmatrix}}_{\text{pt. of intersection}}{}_{(1\times2)}$$

→ Find pt. of intersection of Transformed line

⇓

After Transformation

$$y_i^* = m_1^* x_i^* + b_1^*$$
$$y_i^* = m_2^* x^* + b_2^*$$

Transformed set of lines

m_1^* and m_2^* — Slopes also get Transformed

⇓

Pt. of intersection of these lines through previous method gives us.

$$[x_i^* \quad y_i^*] = \begin{bmatrix} \frac{b_1^* - b_2^*}{m_2^* - m_1^*} & \frac{b_1^* m_2^* - b_2^* m_1^*}{m_2^* - m_1^*} \end{bmatrix}$$

Transformed coordinates.

↳ If we transform them through generalised 2-D transformation matrix.

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$\therefore \; [x^* \;\; y^*] = [x \;\; y] \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left\{ \begin{aligned} x^* &= ax + cy \\ y^* &= bx + dy \end{aligned} \right\} \text{—— (a)}$$

→ Find out b^* in terms of b.

$$y^* = m^* x^* + b^*$$
$$b^* = y - m^* x^* \text{ —— (1)}$$

From (a)

$$b_i^* = (bx + dy) - m^*(ax + cy)$$

$$b_i^* = bx + dy - m^* ax - m^* cy.$$

from (1)

$$b_i^* = bx + dy - \left(\frac{b + dm_i}{a + cm_i}\right)(ax + cy)$$

(Simplify)
$$= bx + dy - \left(\frac{bax + bcy + admx + dmcy}{a + cm}\right)$$

$$b_i^* = \frac{b_i (d - cm_i^*)}{}$$

$b_i \rightarrow y - mx$

$abx + bxcm + ady + dycm - bax - bcy - admx - dmcy.$

$bxcm + ady - bcy - admx.$

$ad(y - mx) - bc(y - mx)$

$$\frac{(y - mx)(ad - bc)}{(a + cm)}$$

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Replacing m_i^*

$$b_1^* = b_i\left[\frac{ad-bc}{a+cm_i}\right] \rightarrow (3)$$

Now we get x_i^* and y_i^* as after substitution (i) & (3)

↓

$$x_i^*, y_i^* \left[\frac{a(b_1-b_2)+c(b_1m_2-b_2m_1)}{m_2-m_1}, \frac{b(b_1-b_2)+d(b_1m_2-b_2m_1)}{m_2-m_1}\right] \quad (4)$$

↓

pts. of intersection of transformed line.

Put & transform original pt. of intersection with $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

⇓

We get same result as (4)

∴ Pts. of intersection are same after transformation.

↓

$$\underset{(1\times 2)}{\left[\frac{b_1-b_2}{m_2-m_1} \quad \frac{b_1m_2-b_2m_1}{m_2-m_1}\right]} \underset{(2\times 2)}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

$$= \underset{(1\times 2)}{\left[\frac{a(b_1-b_2)+c(b_1m_2-b_2m_1)}{m_2-m_1} \quad \frac{b(b_1-b_2)+d(b_1m_2-b_2m_1)}{m_2-m_1}\right]}$$

↓

same as (4)

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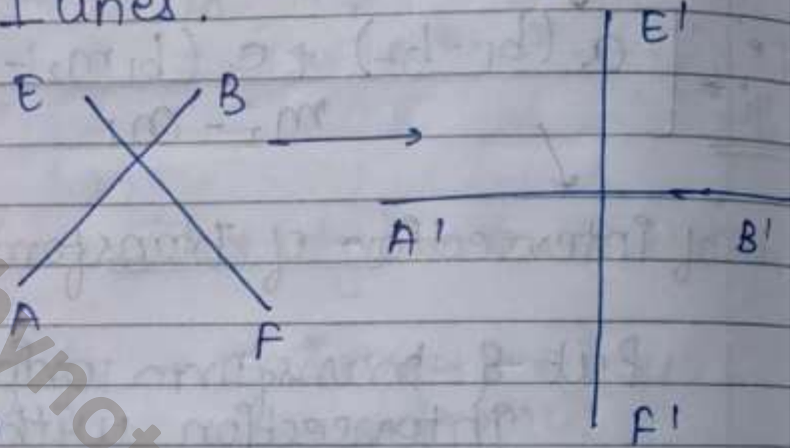
→ Intersecting lines remain intersecting
↓
But now we need to see whether angle intersected is same or not.

(1) A pair of non ⊥ lines get transformed to ⊥ lines.

(2) By inverse T^{-1} a pair of ⊥ lines can get transformed to non ⊥ lines.



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→ Intersecting lines & rigid body Transformations

When do ⊥ lines transform as ⊥ lines?

Consider scalar & vector products of 2 edge vectors
↓

① —— $V_1 \cdot V_2 = V_{1x} V_{2x} + V_{1y} V_{2y}$. $(x_1x_2 + y_1y_2)$

V_{1x}: $(x_2 - x_1)$, V_{1y}: $(y_2 - y_1)$

2 lines l_1 & l_2

$= |V_1| \, |V_2| \cos\theta$

$|V_1|$: magnitude of V_1

② —— $V_1 \times V_2 = (V_{1x} V_{2y} - V_{1y} V_{2x}) \bar{K}$

$(x_1 y_2 - y_1 x_2)$

{unit vector ⊥ to plane containing V_1 & V_2}

$= |V_1| |V_2| \bar{k} \sin\theta.$

Let us now transform these vectors by 2x2 general transformation matrix & find dot & cross product.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} V_{1x} & V_{2x} \\ V_{1y} & V_{2y} \end{bmatrix} = (1).$$

$$\begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} = \begin{bmatrix} aV_{1x} + cV_{1y} & aV_{2x} + cV_{2y} \\ bV_{1x} + dV_{1y} & bV_{2x} + dV_{2y} \end{bmatrix}$$

($aV_{1x} + cV_{1y} = V_{1x}^*$, $bV_{1x} + dV_{1y} = V_{1y}^*$, $aV_{2x} + cV_{2y} = V_{2x}^*$, $bV_{2x} + dV_{2y} = V_{2y}^*$)

$$(a^2+b^2) V_{1x} V_{2x} + (c^2+d^2) V_{1y} V_{2y} + (ac+bd)(V_{1x} V_{2y} + V_{1y} V_{2x})$$

$$= |V_1'| |V_2'| \cos\theta.$$

Find dot & cross prod. of Transformed Vectors

$$V_1^* V_2^* = V_{1x}^* V_{2x}^* + V_{1y}^* V_{2y}^*$$

$$V_1^* \times V_2^* = (V_{1x}^* V_{2y}^* - V_{1y}^* V_{2x}^*) \bar{k}$$

$$V_1^* V_2^* = (aV_{1x} + cV_{1y})(aV_{2x} + cV_{2y}) + (bV_{1x} + dV_{1y})(bV_{2x} + dV_{2y})$$

From (1)

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$a^2 V_{1x} V_{2x} + ac V_{1x} V_{2y} + ac V_{1y} V_{2x}$

$+ c^2 V_{1y} V_{2y} + b^2 V_{1x} V_{2x} +$

$bd V_{1x} V_{2y} + db V_{1y} V_{2x} + d^2 V_{1y} V_{2y}$

(3)

$= (a^2+b^2) V_{1x} V_{2x} + (c^2+d^2) V_{1y} V_{2y}$

$+ (ac+bd)(V_{1x} V_{2y} + V_{2x} V_{1y})$

$= |V_1^*| |V_2^*| \cos\theta$

$V_1^* \times V_2^* = (V_{1x}^* V_{2y}^* - V_{1y}^* V_{2x}^*) \hat{n}$

$= (aV_{1x} + cV_{1y})(bV_{2x} + dV_{2y}) - (bV_{1x} + dV_{1y})(aV_{2x} + cV_{2y})$

$= ab V_{1x} V_{2x} + ad V_{1x} V_{2y} + cb V_{1y} V_{2x} +$

$cd V_{1y} V_{2y} - (ab V_{1x} V_{2x} + bc V_{1x} V_{2y} +$

$da V_{1y} V_{2x} + dc V_{1y} V_{2y})$

$= V_{1x} V_{2y} (ad-bc) + (cb-da) V_{1y} V_{2x}$

$= (V_{1x} V_{2y} - V_{2x} V_{1y})(ad-bc) \hat{n}$

(4)

$= |V_1^*| |V_2^*| \sin\theta \, \hat{n}$

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- If we want ⊥ lines to remain ⊥, then after & before transformation vector & scalar prod. would remain same

∴ (1) must be equal to (3)
and (2) must be equal to (4)

⇓

$a^2+b^2=1$ { equating (1) & (3) }.
$c^2+d^2=1$
$ac+bd=0$

from (2) and (4)

$ad-bc=1$

For a Transformation matrix $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$
then, acc. to this, this matrix must hold all of these above properties so as to make orientation same.

The matrix that holds all these 4 properties is an "orthogonal matrix"

Transformation with this matrix will have ⊥ lines as ⊥ only.

Putting a, b, c, d as rotational matrix.

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$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A pure rotational matrix is an orthogonal matrix

⇓

Thus, pure rotation makes angle same.

$ac + bd = -\sin\theta\cos\theta + \sin\theta\cos\theta = \underline{0}$

$a^2 + b^2 = \sin^2\theta + \cos^2\theta = \underline{1}$

$c^2 + d^2 = \sin^2\theta + \cos^2\theta = \underline{1}$

↓

NOTE:- Transformations where angle remains preserved is

⇓

Solid body / rigid body Transformation matrix.

→ as after pure rotation, shape doesn't change.

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